Generating Hodges' Formula with an $Lw_{1+\infty}$ Ward Identity

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Based on 2506.05460 with Noah Miller and Alfredo Guevara

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$L\mathfrak{w}_{1+\infty}$ in Gravitational Scattering

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This talk will describe a result related to these questions:

Hodges' formula for (stripped) tree-level MHV gravitational scattering amplitudes [Hodges '12], expressed as a *forest formula*, is generated by a one-particle recursion that corresponds to a 2D Ward identity of $\mathrm{L}\mathfrak{w}_{1+\infty}$ symmetry in the celestial dual.

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Next, consider the $n \times n$ matrix Ψ with entries given by

$$\Psi_{ij} = \begin{cases} -\frac{[ij]}{\langle ij\rangle} a_i a_j & i \neq j \\ \sum_{k=1,k\neq i}^{k=n} \frac{[ik]}{\langle ik\rangle} a_i a_k & i = j \end{cases}, \qquad a_i = \langle \alpha i\rangle^2$$

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Hodges' formula describes the "stripped amplitude" M_n as the determinant of an $(n-3)\times(n-3)$ minor of Ψ, $|\Psi|^{123}_{123}$, where rows and columns 1, 2, 3 are removed:

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A slightly more general form of Hodges' formula can also be related to the NSVW tree formula [Nguyen, Spradlin, Volovich, Wen '09].

Outline

1. Main result: new one-particle recursion relation for Hodges' formula
2. Proof of main result using the matrix-tree theorem
3. Main result interpreted as an $L\mathfrak{w}_{1+\infty}$ Ward identity
4. Comments on momentum conservation

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Note that the action of the recursion is simply the action of the exponentiated tower of universal massless soft factors [Hamada, Shiu '18, Li, Lin, Zhang '18, Guevara '19, Bautista, Guevara '19] that generate the action of $\mathfrak{w}_{1+\infty}$ on hard massless particles [Himwich, Pate '21, Adamo, Mason, Sharma '21].

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So, Hodges' (stripped) determinant formula can be obtained by simply acting with the exponentiated soft factor repeatedly on the stripped three-point amplitude.

Recursion involving the exponentiated soft factor, accompanied by the shift of a second particle, was also studied in a celestial context in [Guevara '19, '22, Ren, Schrieber, Sharma, Wang '23].

Other recursive formulae

Our recursive formula is a one-particle (non-momentum-conserving) shift that acts on all of the gravitons, both positive and negative helicity:

$$M_n(1\dots n) = \sum_{i=1}^{n-1} \frac{[ni]}{\langle ni\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} M_{n-1}(1\dots \hat{i}\dots n-1), \qquad |\hat{i}\rangle = |i\rangle, \;\; |\hat{i}] = |i] + \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle}|n]$$

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"Inverse soft recursion" [Boucher-Veronneau, Larkoski '11] is a 2-particle BCFW shift:

$$M_n(1,\dots,n) = \sum_{i=3}^{n} \frac{[1i]}{\langle 1i\rangle} \frac{\langle i2\rangle^2}{\langle 12\rangle^2} M_{n-1}(\hat{2}\dots \hat{i} \dots n)$$

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Hodges' original recursive formula also shifts two particles:

$$M_n(1\ldots n) = \sum_{i=3}^{n-1} \frac{[in]}{\langle in\rangle} \frac{\langle 1i\rangle\langle 2i\rangle}{\langle 1n\rangle\langle 2n\rangle} M_{n-1}(\hat{1}\ldots \hat{i}\ldots n-1)$$

$$|\hat{1}\rangle = |1\rangle, \qquad |\hat{1}] = \frac{(1+n)|i\rangle}{\langle 1i\rangle}, \qquad |\hat{i}\rangle = |i\rangle, \qquad |\hat{i}] = \frac{(i+n)|1\rangle}{\langle i1\rangle}$$

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To prove the recursive formula, we use the *matrix-tree theorem* (following [Feng, He '12]), which implies:

$$|\Psi|^{123}_{123} = \sum_{F\in\mathcal{F}^n_{123}} \left(\prod_{e_{ij}\,\in\,\text{edges}(F)} -\Psi_{ij} \right),$$

where $\mathcal{F}^n_{123}$ is the set of *rooted forests* consisting of three *rooted trees* with roots $1, 2, 3$ and nodes valued in $1, \ldots, n$. A *rooted tree with root k* is a tree containing the node k, so the rooted forests have three disconnected trees with the roots $1, 2, 3$ each in a different tree.

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$$M_n = \langle 12\rangle^8 \left(c_{123} \frac{a_1 \, a_2 \, a_3}{\prod_{\ell=1}^n a_\ell} \right)^2 \sum_{F\in\mathcal{F}^n_{123}} \left(\prod_{e_{ij} \in \text{edges}(F)} \frac{[ij]}{\langle ij\rangle} a_i a_j \right).$$

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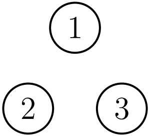

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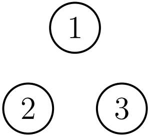

This is the only element of $\mathcal{F}^3_{123}$ (i.e. the set of rooted forests with roots 1,2,3 and total number of nodes $n = 3$) and is the base of the recursion.

Low-point examples: $n = 4$

At four points,

$$M_4 = \left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \frac{1}{\langle \alpha 4\rangle^2} \right)^2 \left(\frac{[14]}{\langle 14\rangle}\langle \alpha 1\rangle^2\langle \alpha 4\rangle^2 + \frac{[24]}{\langle 24\rangle}\langle \alpha 2\rangle^2\langle \alpha 4\rangle^2 + \frac{[34]}{\langle 34\rangle}\langle \alpha 3\rangle^2\langle \alpha 4\rangle^2 \right).$$

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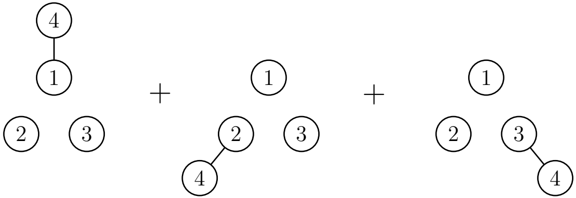
This is represented by the sum over the forests



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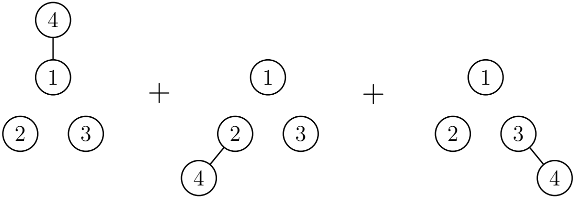

that are the elements of $\mathcal{F}^4_{123}$ (i.e. the set of rooted forests with roots 1,2,3 and total number of nodes $n = 4$). In each forest, the edges carry multiplicative weight factor

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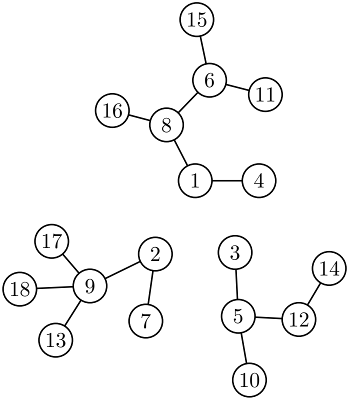
Higher-point example: $n = 18$



This is a rooted forest of three trees with roots 1,2,3. It is an element of $\mathcal{F}^{18}_{123}$, which contributes to M_{18}.

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and the recursive formula

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$$\prod_{j\,\in\,\text{neighbors of }i} \frac{[ij]}{\langle ij\rangle} a_i a_j \;\;\rightarrow\;\; \prod_{j\,\in\,\text{neighbors of }i} \left(\frac{[ij]}{\langle ij\rangle} a_i a_j + \frac{[nj]}{\langle ij\rangle}\frac{\langle \alpha n\rangle}{\langle \alpha i\rangle} a_i a_j \right).$$

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$$\prod_{j \,\in\, \text{neighbors of } i} \frac{[ij]}{\langle ij\rangle} a_i a_j \;\;\to\;\; \prod_{j \,\in\, \text{neighbors of } i} \left(\frac{[ij]}{\langle ij\rangle} a_i a_j + \frac{[nj]}{\langle ij\rangle} \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle} a_i a_j \right).$$

This can be understood using forests with a new node n attached in a special way.

Proof of Recursive Formula

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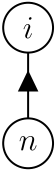
This edge is drawn with an arrow pointing from n to i, with the special weight



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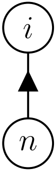

$$= \frac{[in]}{\langle in\rangle}\frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2}$$

The directed arrow on this edge is drawn to anticipate that nodes connected to n will have another special weight.

Proof of Recursive Formula (continued)

2. Consider the neighbors of a node i. The transformation

$$\prod_{j \in \text{neighbors of } i} \frac{[ij]}{\langle ij \rangle} a_i a_j \quad \rightarrow \prod_{j \in \text{neighbors of } i} \left(\frac{[ij]}{\langle ij \rangle} a_i a_j + \frac{[nj]}{\langle ij \rangle} \frac{\langle \alpha n \rangle}{\langle \alpha i \rangle} a_i a_j \right)$$

(when expanded) involves a sum over all combinations of trees where each node j formerly connected to i instead either has one of two options:

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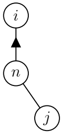
This gives a product of special edges e_{in} and e_{jn} (recall $a_i = \langle \alpha i\rangle$)



$$= \underbrace{\left(\frac{[nj]}{\langle ij\rangle} \langle \alpha i\rangle^2 \langle \alpha j\rangle^2 \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle} \right)}_{} \underbrace{\left(\frac{[in]}{\langle in\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} \right)}_{}$$

part

part

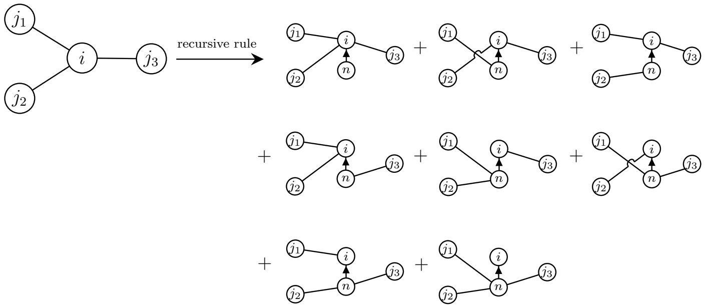
The highlighted factors emphasize that the special weight of e_{jn} generated by the recursive formula *depends on* i because of the directed arrow edge e_{in}.

Proof of Recursive Formula (continued)

The recursive transformation

$$\prod_{j \in \text{neighbors of } i} \frac{[ij]}{\langle ij \rangle} a_i a_j \quad \to \quad \prod_{j \in \text{neighbors of } i} \left(\frac{[ij]}{\langle ij \rangle} a_i a_j + \frac{[nj]}{\langle ij \rangle} \frac{\langle \alpha n \rangle}{\langle \alpha i \rangle} a_i a_j \right)$$

thus has graphical form



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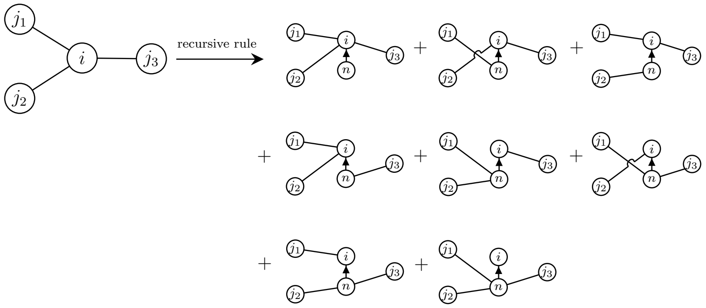

Generally, if a collection of s neighbors attach to the node i in a graph, the recursive formula transforms this into a sum of 2^s graphs where n attaches to i by an arrowed edge and all s neighbors can attach to either i or n. In this example $s = 3$.

Proof of Recursive Formula (continued)

The full result of the recursive formula, summed over i, transforms the sum over all rooted forests $\mathcal{F}_{123}^{n-1}$ with $n-1$ nodes into a sum over i of all *arrowed rooted forests* with n nodes, i.e. rooted forests with n nodes in which the node n is special and points to exactly one neighbor node i with an arrowed edge.

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We denote the set of arrowed rooted forests by $\mathcal{F}_{123}^{n\to i}$.

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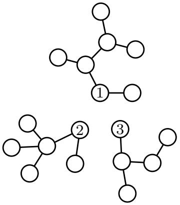
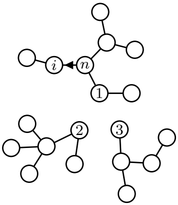
Schematically, the action of the recursive formula takes a sum over forests in $\mathcal{F}_{123}^{n-1}$ to a sum over forests in $\mathcal{F}_{123}^{n\to i}$:

$$M_{n-1} \propto \sum_{\mathcal{F}_{123}^{n-1}}$$



$$\xrightarrow[\text{formula}]{\text{recursive}} \sum_{i=1}^{n-1} \sum_{\mathcal{F}_{123}^{n\to i}}$$



Proof of Recursive Formula (continued)

Using the notation of arrowed rooted forests, we now have

$$\sum_{i=1}^{n-1} \frac{[ni]}{\langle ni\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} M_{n-1}(\ldots, |i\rangle, |i] + \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle}|n], \ldots)$$

$$= \langle 12\rangle^8 \sum_{i=1}^{n-1} \left(c_{123} \frac{a_1\, a_2\, a_3}{\prod_{\ell=1}^{n-1} a_\ell} \right)^2 \sum_{F \in \mathcal{F}_{123}^{n\to i}} \left(\prod_{e_{kl} \in \text{edges}(F)} w_{kl} \right),$$

where, assuming without loss of generality that $k < l$, the weights w_{kl} are defined by the edges we described above:

$$w_{kl} = \begin{cases} \frac{[kl]}{\langle kl\rangle} a_k a_l, & k, l \neq n, \\ \frac{[in]}{\langle in\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2}, & k = i, l = n, \\ \frac{[nk]}{\langle ik\rangle} \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle} a_i a_k, & k \neq i, l = n. \end{cases}$$

Proof of Recursive Formula (contintued)

Comparing the action of the recursion

$$\sum_{i=1}^{n-1} \frac{[ni]}{\langle ni\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} M_{n-1}(\ldots, |i\rangle, |i] + \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle}|n], \ldots)$$

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and the n-point formula

$$M_n = \langle 12\rangle^8 \left(c_{123} \frac{a_1\, a_2\, a_3}{\prod_{\ell=1}^{n} a_\ell} \right)^2 \sum_{F\in \mathcal{F}_{123}^{n}} \left(\prod_{e_{ij}\,\in\, \text{edges}(F)} \frac{[ij]}{\langle ij\rangle} a_i a_j \right),$$

we see that the recursive formula holds if we prove that the sum over $i \in [1, n-1]$ of arrowed rooted forests $\mathcal{F}_{123}^{n\rightarrow i}$ appearing in the recursive formula is a factor of $\frac{1}{a_n^2}$ times the sum of regular rooted forests $\mathcal{F}_{123}^{n}$ with regular edges.

Proof of Recursive Formula (contintued)

Comparing the action of the recursion

$$\sum_{i=1}^{n-1} \frac{[ni]}{\langle ni\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} M_{n-1}(\ldots, |i\rangle, |i] + \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle}|n], \ldots)$$

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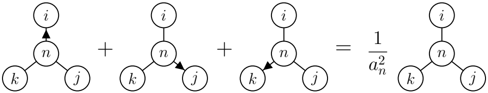
To conclude, we want to prove that the sum over $i \in [1, n-1]$ of arrowed forests amounts to forgetting the decoration of the arrow (and multiplying by a factor of $\frac{1}{a_n^2}$).

Proof of Recursive Formula (concluded)

Consider an example of summing the placement of an arrowed edge over three neighbors of n. This is equal to a graph with the arrow removed multiplied by $\frac{1}{a_n^2}$.



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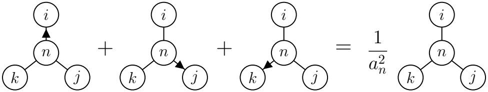

Generally, let $\underline{m}$ denote the set of neighbors of n, and $\iota \in \underline{m}$ index this set. We show

$$\sum_{\iota \in \underline{m}} \left(\prod_{\iota \in \underline{m}} w_{\iota n} \right) = \frac{1}{a_n^2} \left(\prod_{\iota \in \underline{m}} \frac{[\iota n]}{\langle \iota n \rangle} a_\iota a_n \right),$$

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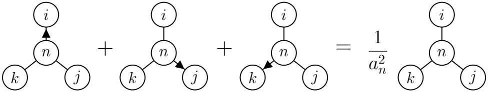

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which largely amounts to use of the partial fraction identity

$$\sum_{\iota=1}^{m} \left(\frac{\langle \alpha \iota \rangle}{\langle \alpha n \rangle} \right)^{m-1} \frac{1}{\langle \iota n \rangle} \prod_{\substack{j=1 \\ j \neq \iota}}^{m} \frac{1}{\langle \iota j \rangle} = - \prod_{\iota=1}^{m} \frac{1}{\langle n \iota \rangle}.$$

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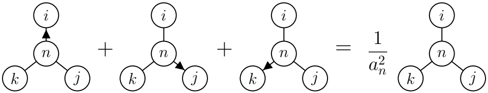

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Performing the set of all such sums over all such possible arrowed rooted forests turns out to be exactly the sum over the set of all regular (unarrowed) rooted forests multiplied by $\frac{1}{a_n^2}$, which concludes the proof.

Recursive Formula as an $Lw_{1+\infty}$ Ward identity

Recall the recursive formula

$$M_n = \sum_{i=1}^{n-1} \frac{[ni]}{\langle ni\rangle} \frac{\langle \alpha i\rangle^2}{\langle \alpha n\rangle^2} M_{n-1}(\,|1\rangle\,, |1], \cdots, |i\rangle, |i] + \frac{\langle \alpha n\rangle}{\langle \alpha i\rangle}|n], \cdots, |n{-}1\rangle\,, |n{-}1]\,).$$

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To interpret this as an $L\mathfrak{w}_{1+\infty}$ Ward identity, we will show that for particular choices of spinors, the resursive action follows from the action of a $L\mathfrak{w}_{1+\infty}$ current with OPE

$$G^+(z_i, \tilde\lambda_i)\, G^+(z_j, \tilde\lambda_j) \sim \frac{[ij]}{z_{ij}} G^+(z_j, \tilde\lambda_j + \tilde\lambda_i),$$

where $\tilde\lambda_i$ is interpreted as color index for the $\mathfrak{w}_{1+\infty}$ symmetry ($G^+ G^-$ is analogous).

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This is related by a change of basis of the $\mathfrak{w}_{1+\infty}$ algebra to an OPE

$$w^p_m(z_1) w^q_n(z_2) \sim \frac{1}{z_{12}} \left(m(q-1) - n(p-1)\right) w^{p+q-2}_{m+n}(z_2).$$

Review of Holomorphic Collinear Limits

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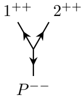
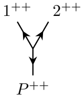
The tree-level graviton splitting functions are, with $p_P = p_1 + p_2$:

$\text{Split}^{+2,+2}_{+2}$:



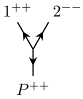
$$\frac{[12]}{\langle 12\rangle}\,\frac{\langle P\alpha\rangle^4}{\langle 1\alpha\rangle^2\langle 2\alpha\rangle^2}$$

$\text{Split}^{+2,+2}_{-2}$:



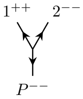
$$0$$

$\text{Split}^{+2,-2}_{-2}$:



$$\frac{[12]}{\langle 12\rangle}\,\frac{\langle 2\alpha\rangle^6}{\langle P\alpha\rangle^4\langle 1\alpha\rangle^2}$$

$\text{Split}^{+2,-2}_{+2}$:



$$\frac{\langle 12\rangle}{[12]}\,\frac{\langle \alpha P\rangle^4\langle \alpha 2\rangle^2}{\langle \alpha 1\rangle^6}\,.$$

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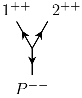	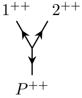	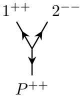	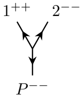
$\frac{[12]}{\langle 12\rangle}\frac{\langle P\alpha\rangle^4}{\langle 1\alpha\rangle^2\langle 2\alpha\rangle^2}$	0	$\frac{[12]}{\langle 12\rangle}\frac{\langle 2\alpha\rangle^6}{\langle P\alpha\rangle^4\langle 1\alpha\rangle^2}$	$\frac{\langle 12\rangle}{[12]}\frac{\langle \alpha P\rangle^4\langle \alpha 2\rangle^2}{\langle \alpha 1\rangle^6}$.

With the choice $p_i^\mu = \frac{\omega_i}{2}(1 + z_i\bar{z}_i,\ z_i + \bar{z}_i,\ -i(z_i - \bar{z}_i),\ 1 - z_i\bar{z}_i)$,

$$\lambda_i = \begin{pmatrix} 1 \\ z_i \end{pmatrix}, \quad \tilde{\lambda}_i = \begin{pmatrix} \omega_i \\ \omega_i \bar{z}_i \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \langle ij\rangle = z_{ij}, \quad [ij] = \omega_i\omega_j\bar{z}_{ij}, \quad \langle \alpha i\rangle = 1,$$

then the collinear splitting functions read

$$\lim_{z_{ij}\to 0} G^+(z_i, \tilde{\lambda}_i)\, G^\pm(z_j, \tilde{\lambda}_j) = \frac{[ij]}{z_{ij}} G^\pm(z_j, \tilde{\lambda}_j + \tilde{\lambda}_i).$$

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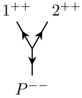
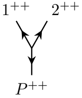
The tree-level graviton splitting functions are, with $p_P = p_1 + p_2$:

$\mathrm{Split}^{+2,+2}_{+2}$:



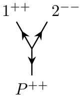
$$\frac{[12]}{\langle 12\rangle}\frac{\langle P\alpha\rangle^4}{\langle 1\alpha\rangle^2\langle 2\alpha\rangle^2}$$

$\mathrm{Split}^{+2,+2}_{-2}$:



$$0$$

$\mathrm{Split}^{+2,-2}_{-2}$:



$$\frac{[12]}{\langle 12\rangle}\frac{\langle 2\alpha\rangle^6}{\langle P\alpha\rangle^4\langle 1\alpha\rangle^2}$$

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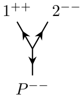

$$\frac{\langle 12\rangle}{[12]}\frac{\langle\alpha P\rangle^4\langle\alpha 2\rangle^2}{\langle\alpha 1\rangle^6}.$$

With the choice $p_i^\mu = \frac{\omega_i}{2}(1 + z_i\bar{z}_i,\ z_i + \bar{z}_i,\ -i(z_i - \bar{z}_i),\ 1 - z_i\bar{z}_i)$,

$$\lambda_i = \begin{pmatrix}1\\ z_i\end{pmatrix},\quad \tilde{\lambda}_i = \begin{pmatrix}\omega_i\\ \omega_i\bar{z}_i\end{pmatrix},\quad \alpha = \begin{pmatrix}0\\ 1\end{pmatrix},\qquad \langle ij\rangle = z_{ij},\quad [ij] = \omega_i\omega_j\bar{z}_{ij},\quad \langle\alpha i\rangle = 1,$$

then the collinear splitting functions read

$$\lim_{z_{ij}\to 0} G^+(z_i,\tilde{\lambda}_i)\, G^\pm(z_j,\tilde{\lambda}_j) = \frac{[ij]}{z_{ij}}\, G^\pm(z_j,\tilde{\lambda}_j + \tilde{\lambda}_i).$$

The $\tilde{\lambda}_i$ is interpreted as color index for the $\mathfrak{w}_{1+\infty}$ symmetry.

Review of $\mathfrak{w}_{1+\infty}$ in Two Bases

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Abstractly, the $\mathfrak{w}_{1+\infty}$ algebra is just the algebra of functions of two variables (say q and p) where the Lie bracket is the Poisson bracket.

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Two natural bases for the Lie algebra elements are 2D plane waves and monomials:

$$\mathbf{T}_{\tilde{\lambda}} \equiv \exp\left(i\,\mathrm{q}\,\tilde{\lambda}^{\dot{1}} + i\,\mathrm{p}\,\tilde{\lambda}^{\dot{2}}\right), \qquad [\mathbf{T}_{\tilde{\lambda}_1}, \mathbf{T}_{\tilde{\lambda}_2}] = [12]\mathbf{T}_{\tilde{\lambda}_1+\tilde{\lambda}_2},$$

$$\mathbf{T}^p_m \equiv \frac{1}{2}\mathrm{q}^{p+m-1}\mathrm{p}^{p-m-1}, \qquad [\mathbf{T}^p_m, \mathbf{T}^q_n] = \left(m(q-1) - n(p-1)\right)\mathbf{T}^{p+q-2}_{m+n},$$

where we recall $[ij] = \tilde{\lambda}^{\dot{2}}_i \tilde{\lambda}^{\dot{1}}_j - \tilde{\lambda}^{\dot{1}}_i \tilde{\lambda}^{\dot{2}}_j$.

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where we recall $[ij] = \tilde{\lambda}^{\dot{2}}_i \tilde{\lambda}^{\dot{1}}_j - \tilde{\lambda}^{\dot{1}}_i \tilde{\lambda}^{\dot{2}}_j$. The first basis corresponds to the momentum space particles $G^+(z, \tilde{\lambda})$ and the second basis the conformal primary modes $w^p_m(z)$:

$$\lim_{z_{ij}\to 0} G^+(z_i, \tilde{\lambda}_i)\, G^{\pm}(z_j, \tilde{\lambda}_j) = \frac{[ij]}{z_{ij}} G^{\pm}(z_j, \tilde{\lambda}_j + \tilde{\lambda}_i),$$

$$\lim_{z_{12}\to 0} w^p_m(z_1) w^q_n(z_2) = \frac{1}{z_{12}} (m(q-1) - n(p-1))\, w^{p+q-2}_{m+n}(z_2).$$

Review of $\mathfrak{w}_{1+\infty}$ in Two Bases (continued)

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$$\begin{aligned}&\mathrm{q}^{p+m-1}\mathrm{p}^{p-m-1}\\ &= i^{2-2p}(p+m-1)!(p-m-1)!\lim_{\epsilon\to 0}\epsilon\oint\frac{d\bar{z}}{2\pi i}\frac{1}{\bar{z}^{p-m}}\int_0^\infty\frac{d\omega}{\omega}\,\omega^{2-2p+\epsilon}\,e^{i\omega(\mathrm{q}+\mathrm{p}\bar{z})}.\end{aligned}$$

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Likewise, the usual $w^p_m(z)$ currents are also related to $G^+(z,\tilde{\lambda})$ by

$$w^p_m(z) = \frac{1}{2} i^{2-2p}(p+m-1)!(p-m-1)! \lim_{\epsilon\to 0} \epsilon \oint \frac{d\bar{z}}{2\pi i}\frac{1}{\bar{z}^{p-m}} \int_0^\infty \frac{d\omega}{\omega}\,\omega^{2-2p+\epsilon}\, G^+(z,\tilde{\lambda}).$$

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$$\left\langle G^{-}(z_1, \tilde{\lambda}_1)\, G^{-}(z_2, \tilde{\lambda}_2)\, G^{+}(z_3, \tilde{\lambda}_3)\right\rangle = \frac{z_{12}^8}{z_{12}^2 z_{23}^2 z_{31}^2}.$$

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From the recursive formula we have proved (in the spinor choices just described) any correlator that satisfies these two rules immediately satisfies

$$M_n = \left\langle G^-(z_1, \tilde{\lambda}_1)\, G^-(z_2, \tilde{\lambda}_2) \dots G^+(z_n, \tilde{\lambda}_n) \right\rangle,$$

which is how the $L\mathfrak{w}_{1+\infty}$ Ward identity generates Hodges' formula.

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  In the language of celestial OPEs: the leading term in OPEs derived from holomorphic collinear limits is covariant under a chiral half of Poincaré involving two translations, but it is not covariant on its own under the other two translations.

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- The recursive formula is the action of the exponential universal massless soft factor [Hamada, Shiu '18, Li, Lin Zhang '18, Guevara '19, Bautista, Guevara '19].
  This generates the action of $\mathfrak{w}_{1+\infty}$ on hard particles [Himwich, Pate '21, Adamo, Mason, Sharma '21] but does not conserve momentum, so the most it can generate is a stripped amplitude.

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From the base case $M_3 = \left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\right)^2$, Hodges' formula for the stripped tree-level mostly-plus MHV graviton amplitude M_n with $n > 3$ is generated by the one-particle recursion relation

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3. Is there an analogous recursion relation for all-plus amplitudes?

   From the realization of $L\mathfrak{w}_{1+\infty}$ in twistor space, one natural expectation is that $L\mathfrak{w}_{1+\infty}$ generates amplitudes of the self-dual sector. Self-dual amplitudes vanish at tree level, but the recursion could appear at one loop, where it is known that $L\mathfrak{w}_{1+\infty}$ symmetry is perturbatively exact [Ball, Narayanan, Saltzer, Strominger '21, Bittleston '22].